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Journal of Algebra 266 (2003) 77–86

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Weak projectives of finite semigroups

Yevhen Zelenyuk

Faculty of Cybernetics, Kyiv Taras Shevchenko University, vul. Glushkova 2, korp. 6, 03680, Kyiv, Ukraine

Received 30 November 2001

Communicated by T.E. Hall

Abstract

It is described weak projectives in the category of finite semigroups. These are precisely finite weak projectives in the category of compact right topological semigroups.

Keywords: Projective; Absolute coretract; Finite semigroup; Compact right topological semigroup

In [4], P.G. Trotter characterized projectives in the category $\mathcal{FR}$ of finite regular semigroups. In particular, he proved that projectives in $\mathcal{FR}$ are bands. In the category $\mathcal{F}$ of finite semigroups, there are no projectives [3]. The aim of this paper is to describe weak projectives in $\mathcal{F}$. An object S in some category is a *projective* (a *weak projective*) if for every morphism $f : S \to Q$ and every epimorphism (surjective epimorphism) $g : T \to Q$ there exists a morphism $h : S \to T$ such that $g \circ h = f$.

The author came to the weak projectives in $\mathcal{F}$ from the finite weak absolute coretracts in the category $\mathcal{C}$ of compact right topological semigroups (morphisms in $\mathcal{C}$ are continuous homomorphisms). A semigroup endowed with a topology is *right topological* if all its right shifts are continuous. A significant example of compact right topological semigroup is the Stone–Čech compactification βS of a discrete semigroup S (see [1]). An object S in some category is an *absolute coretract* (a *weak absolute coretract*) if for every epimorphism (surjective epimorphism) $f : T \to S$ there exists a morphism $g : S \to T$ such that $f \circ g = \mathrm{id}_S$. The weak absolute coretracts in $\mathcal{C}$ arose in solving the following two questions.

E-mail addresses: grishko@i.com.ua, zelenyuk@unicyb.kiev.ua.
URL address: http://www.i.com.ua/~grishko/zelenyuk.html.

0021-8693/03/$ – see front matter
doi:10.1016/S0021-8693(03)00278-3

The first of them is concerned with topological groups. For every topological group (G,τ), the subset τ^* of all nonprincipal ultrafilters on G converging to the identity in the topology τ is a closed subsemigroup in βG which is called a semigroup of ultrafilters of (G,τ). Which finite semigroups can be semigroups of ultrafilters of topological groups? It turns out that, under Continuum Hypothesis, for each finite weak absolute coretract C in $\mathcal{C}$, there is a group topology τ on countable Boolean group with τ^* isomorphic to C, and that each finite semigroup of ultrafilters of a countable topological group is an idempotent weak absolute coretract in $\mathcal{F}$ [8]. (Observe that each countable topological group with finite semigroups of ultrafilters contains an open Boolean subgroup and cannot be constructed without additional set-theoretic assumptions [5,6].)

The second question is concerned with the semigroup $\beta\mathbb{N}$. Until now it is unknown whether there are elements of finite order in $\beta\mathbb{N}$ other than idempotents. In [7] it was proved that there are no non-trivial finite groups in $\beta\mathbb{N}$ (see also [1, Section 7.1]). Which finite bands exist in $\beta\mathbb{N}$? It is well known that $\beta\mathbb{N}$ contains closed subsemigroups admitting a continuous homomorphism onto any finite semigroup. Hence, $\beta\mathbb{N}$ contains isomorphic copies of any finite weak absolute coretract in $\mathcal{C}$.

In [9] author described finite idempotent weak absolute coretracts in $\mathcal{C}$ and proved that these are precisely idempotent weak absolute coretracts in $\mathcal{F}$. In this paper we prove the following Main Theorem on the collection $\mathcal{P}$ of finite bands from [9].

Main Theorem. *Let S be a finite semigroup. The following statements are equivalent*:

(1) *S is isomorphic to some semigroup of $\mathcal{P}$*;
(2) *S is a projective in $\mathcal{FR}$*;
(3) *S is a weak projective in $\mathcal{F}$*;
(4) *S is a weak absolute coretract in $\mathcal{F}$*;
(5) *S is an absolute coretract in $\mathcal{F}$*;
(6) *S is a weak projective in $\mathcal{C}$*;
(7) *S is a weak absolute coretract in $\mathcal{C}$*;
(8) *S is an absolute coretract in $\mathcal{C}$.*

We prove the Main Theorem by circuits (1) $\Rightarrow$ (6) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (1). The implications (8) $\Rightarrow$ (5) $\Rightarrow$ (4), (6) $\Rightarrow$ (7), and (6) $\Rightarrow$ (3) are trivial. Since, by Hall's theorem, epimorphisms in $\mathcal{FR}$ are surjective (see [4, Theorem 2.1]), (3) $\Rightarrow$ (2) is also clear. So we need to prove (1) $\Rightarrow$ (6), (2) $\Rightarrow$ (1), (4) $\Rightarrow$ (1), and (7) $\Rightarrow$ (8).

We begin with the construction of the collection $\mathcal{P}$.

Denote by U the semigroup of words of the form $i_1 i_2 \cdots i_p \lambda_p \lambda_{p-1} \cdots \lambda_1$, where $i_q, \lambda_q \in \omega$, $1 \leqslant q \leqslant p < \omega$, with the operation

$$i_1 \cdots i_p \lambda_p \cdots \lambda_1 \cdot j_1 \cdots j_q \rho_q \cdots \rho_1 = \begin{cases} i_1 \cdots i_p \rho_p \cdots \rho_1 & \text{if } p = q, \\ i_1 \cdots i_p \lambda_p \cdots \lambda_{q+1} \rho_q \cdots \rho_1 & \text{if } p > q, \\ i_1 \cdots i_p j_{p+1} \cdots j_q \rho_q \cdots \rho_1 & \text{if } p < q. \end{cases}$$

For every $p \in \mathbb{N}$, denote by U_p the subsemigroup of U of words of length $2p$. The semigroup U is a band decomposing into the decreasing chain of its rectangular components U_p. For every subsemigroup S of U, put $S_p = S \cap U_p$.

For every $p \in \mathbb{N}$, $q \in [1, p]$ and $u = i_1 \cdots i_p \lambda_p \cdots \lambda_1 \in U_p$, put $u' = i_1 \cdots i_p$, $u'' = \lambda_p \cdots \lambda_1$, $u'_q = i_q$, and $u''_q = \lambda_q$.

Denote by $\mathcal{P}$ the collection of finite subsemigroups S of U satisfying the following conditions for every $p \in \mathbb{N}$:

(i) if $u \in S_p$, both $u'_p \neq 0$ and $u''_p \neq 0$;
(ii) if $u \in S_p$ and $u'_q \neq 0$ for some $q \in [1, p-1]$, there exists $v \in S_q$ such that v' is the initial segment of u'; and dually, if $u \in S_p$ and $u''_q \neq 0$ for some $q \in [1, p-1]$, there exists $v \in S_q$ such that v'' is the final segment of u'';
(iii) either $u'_p = 1$ for all $u \in S_q$ with $q \geqslant p$ or $u''_q = 1$ for all $u \in S_q$ with $q \geqslant p$.

We indicate also a complete system of non-isomorphic representatives of $\mathcal{P}$.

Denote by $\mathcal{M}$ the set of all matrices $M = (m_{p,q})_{l\times l}$ without the main diagonal $(m_{p,p})$, where $l \in \mathbb{N}$ and $m_{p,q} \in \omega$, satisfying the following conditions for every $p \in [1, l]$:

(a) $m_{0,p} \leqslant m_{1,p} \leqslant \cdots \leqslant m_{p-1,p} \in \mathbb{N}$ and $m_{p,0} \leqslant m_{p,1} \leqslant \cdots \leqslant m_{p,p-1} \in \mathbb{N}$;
(b) either $m_{p-1,p} = 1$ and $m_{p-1,p+1} = \cdots = m_{p-1,l} = 0$ or $m_{p,p-1} = 1$ and $m_{p+1,p-1} = \cdots = m_{l,p-1} = 0$.

For each $M = (m_{p,q})_{l\times l} \in \mathcal{M}$, denote by $S(M)$ the subsemigroup of $\bigcup_{p=1}^{l} U_p$ consisting of all $u \in U_p$, $p \in [1, l]$, which are satisfying the following conditions:

(1) both $u'_p \neq 0$ and $u''_p \neq 0$;
(2) for every $q < r \leqslant p$, if $u'_t = 0$ for all $t \in [q+1, r-1]$, then $u'_r \leqslant m_{q,r}$, and dually, if $u''_t = 0$ for all $t \in [q+1, r-1]$, then $u''_r \leqslant m_{r,q}$.

It is obvious that for every $M \in \mathcal{M}$, $S(M) \in \mathcal{P}$. We claim that every $S \in \mathcal{P}$ is isomorphic to $S(M)$ for some $M \in \mathcal{M}$.

Indeed, let $l = \max\{p\colon\ S_p \neq \emptyset\}$ and for every $p \in [1, l]$, let $I_p = \{u'_p\colon\ u \in S_p\}$ and $\Lambda_p = \{u''_p\colon\ u \in S_p\}$. For every $q < p \leqslant l$, let $I_{q,p}$ be the set of all $i \in I_p$ such that there exists $u \in S_p$ with $u'_p = i$ and $u'_r = 0$ for all $r \in [q+1, p-1]$ and let $m_{q,p} = |I_{q,p}|$. Analogously, for every $q < p \leqslant l$, let $\Lambda_{p,q}$ be the set of all $\lambda \in \Lambda_p$ such that there exists $u \in S_p$ with $u''_p = \lambda$ and $u''_r = 0$ for all $r \in [q+1, p-1]$ and let $m_{p,q} = |\Lambda_{p,q}|$. For every $p \in [1, l]$, choose bijections $f_p : I_p \to \{1, \ldots, m_{p-1,p}\}$ and $g_p : \Lambda_p \to \{1, \ldots, m_{p,p-1}\}$ such that whenever $q < r$, $i \in I_{q,p}$, and $j \in I_{r,p} \setminus I_{q,p}$, one has $f(i) < f(j)$, and dually, whenever $q < r$, $\lambda \in \Lambda_{p,q}$, and $\mu \in \Lambda_{p,r} \setminus \Lambda_{p,q}$, one has $g(\lambda) < g(\mu)$. An easy check shows that $M = (m_{p,q})_{l\times l} \in \mathcal{M}$ and that

$$S \ni i_1 \cdots i_p \lambda_p \cdots \lambda_1 \mapsto f_1(i_1) \cdots f_p(i_p) g_p(\lambda_p) \cdots g_1(\lambda_1) \in S(M)$$

is the required isomorphism.

Let now $M = (m_{p,q})_{l\times l} \in \mathcal{M}$ and $S = S(M)$. For every $q < p \leqslant l$, define families $\mathcal{F}_{q,p}$ and $\mathcal{F}_{p,q}$ of subsets of S_p downstairs induction by q putting

$$\mathcal{F}_{q,p} = \left\{ S_q u S_p : u \in S_p \setminus \bigcup_{r=q+1}^{p-1} S_{r,p} \right\} \quad \text{and} \quad \mathcal{F}_{p,q} = \left\{ S_p u S_q : u \in S_p \setminus \bigcup_{r=q+1}^{p-1} S_{p,r} \right\},$$

where

$$S_{r,p} = \bigcup \mathcal{F}_{r,p} \quad \text{and} \quad S_{p,r} = \bigcup \mathcal{F}_{p,r}.$$

It is easy to see that

$$m_{q,p} = |\mathcal{F}_{q,p}| \quad \text{and} \quad m_{p,q} = |\mathcal{F}_{p,q}|.$$

Consequently, M uniquely determined by S.

The following theorem is the implication (1) $\Rightarrow$ (6). Its coretract version was proved in [9].

Theorem 1. *Every semigroup of $\mathcal{P}$ is a weak projective in $\mathcal{C}$.*

Proof. Let $S \in \mathcal{P}$, let $f : S \to Q$ be a homomorphism, and let $g : T \to Q$ be a surjective continuous homomorphism. We adjoin the identities $\emptyset$, 1_Q, 1_T to S, Q, T, respectively, and extend f, g in the obvious way. We shall inductively construct the homomorphism $h : S \to T$ such that $g \circ h = f$.

Put $S_0 = \{\emptyset\}$, $S_0^p = \bigcup_{q=0}^p S_q$,

$$e_p = \begin{cases} \emptyset & \text{if } p = 0, \\ 1\cdots 11\cdots 1 \in S_p & \text{otherwise.} \end{cases}$$

For every $u \in S_p$, put

$$\dot{u} = \begin{cases} \emptyset \quad \text{if } u \in S_0^1 \text{ or } u'_{p-1} = \cdots = u'_1 = 0, \\ u'_1 \cdots u'_q 1 \cdots 1 \in S_q, \quad \text{where } q = \max\{r < p\colon\ u'_r \neq 0\}, \quad \text{otherwise,} \end{cases}$$

$$\ddot{u} = \begin{cases} \emptyset \quad \text{if } u \in S_0^1 \text{ or } u''_{p-1} = \cdots = u''_1 = 0, \\ 1 \cdots 1 u''_q \cdots u''_1 \in S_q, \quad \text{where } q = \max\{r < p\colon\ u''_r \neq 0\}, \quad \text{otherwise.} \end{cases}$$

Lemma 1.

(a) *For every $u \in S_p$, $\dot{u}u = u\ddot{u} = u$.*
(b) *If $q < p$, $u \in S_p$, and $v \in S_q$, then $\ddot{(uv)} = \ddot{u}v$ and $\dot{(vu)} = v\dot{u}$.*
(c) *For every $u, v \in S_p$, $\ddot{u}\dot{v} = e_{p-1}$.*

The proof of Lemma 1 is an easy check [1].

For every $u \in S_p$, we also put $R(u') = \{v \in S_p\colon\ v' = u'\}$ and $L(u'') = \{v \in S_p\colon v'' = u''\}$. Observe that these are respectively minimal right and minimal left ideals in S_p containing u.

We shall use the fact that every compact right topological semigroup has the smallest ideal which is a completely simple semigroup (see [1, Section 2.2]).

Define h on S_0 by $h(\emptyset) = 1_T$. Suppose that h has been defined on S_0^{p-1}. We shall show that h can be extended to S_p.

Let $I_p = \{u'_p\colon\ u \in S_p\}$ and for every $u \in S_p$, let $\mu(u) = \min\{q < p\colon\ u'_{q+1} = \cdots = u'_{p-1} = 0\}$. For each $i \in I_p$, we choose $w_i \in S_p$ such that $(w_i)'_p = i$ and $\mu(w_i) = \min\{\mu(u)\colon\ u \in S_p$ with $u'_p = i\}$, and then choose a minimal right ideal $R_p(i)$ in $g^{-1}(f(S_p))$ with $g(R_p(i)) \subseteq f(R((w_i)'))$. We observe that, for any $u \in S_p$ with $u'_p = i$, we have $\dot{u}R((w_i)') \subseteq R(u')$, and so $g(h(\dot{u})R_p(i)) \subseteq f(\dot{u})f(R((w_i)')) \subseteq f(R(u'))$. Hence, for any $u \in S_p$, we have $g(h(\dot{u})R_p(u'_p)) \subseteq f(R(u'))$. We define minimal left ideals $L_p(\lambda)$ in $g^{-1}(f(S_p))$ in the dual way. For every $u \in S_p$, we define $h(u)$ to be the idempotent of the group $h(\dot{u})R_p(u'_p)L_p(u''_p)h(\ddot{u})$. Then $gh(u) = f(u)$, because

$$gh(u) \in g\big(h(\dot{u})R_p\big(u'_p\big)\big)g\big(L_p\big(u''_p\big)h(\ddot{u})\big) \subseteq f\big(R(u')\big)f\big(L(u'')\big) = f(\{u\}).$$

Let $v \in S_0^{p-1}$. We shall show that $h(u)h(v) = h(uv)$.

We have $h(u)h(v) \in h(\dot{u})R_p(u'_p)L_p(u''_p)h(\ddot{u})h(v)$. We have also $\dot{u} = (\dot{uv})$, $u'_p = (uv)'_p$, $u''_p = (uv)''_p$, and $h(\ddot{u})h(y) = h(\ddot{u}v) = h((\ddot{uv}))$. So $h(u)h(v)$ and $h(uv)$ belong to the same group in $g^{-1}(f(S_p))$. It will therefore be sufficient to show that $h(u)h(v)$ is idempotent. To establish this, we shall show that $h(u)h(v)h(u) = h(u)$.

We write $h(u) = h(\dot{u})wh(\ddot{u})$ for some $w \in R_p(u'_p)L_p(u''_p)$. Then

$$h(u)h(v)h(u) = h(\dot{u})wh(\ddot{u})h(v)h(\dot{u})wh(\ddot{u}) = h(\dot{u})wh(\ddot{u}v\dot{u})wh(\ddot{u}).$$

Since $\ddot{u}v\dot{u} = (\ddot{uv})\dot{u} = e_{p-1} = \ddot{u}\dot{u}$, then

$$h(u)h(v)h(u) = h(\dot{u})wh(\ddot{u}\dot{u})wh(\ddot{u}) = h(\dot{u})wh(\ddot{u})h(\dot{u})wh(\ddot{u}) = h(u)h(u) = h(u).$$

This establishes that $h(uv) = h(u)h(v)$. Similarly, $h(vu) = h(v)h(u)$.

Let now $v \in S_p$. Again, we have $h(u)h(v) \in h(\dot{u})R_p(u'_p)L_p(v''_p)h(\ddot{v})$ and we have also $\dot{u} = \dot{uv}$, $u'_p = (uv)'_p$, $v''_p = (uv)''_p$, and $\ddot{v} = (\ddot{uv})$. So $h(u)h(v)$ and $h(uv)$ belong to the same group. We shall again show that $h(u)h(v)$ is idempotent by proving that $h(u)h(v)h(u) = h(u)$.

We know that either $w''_p = 1$ for all $w \in S_p$ or $w'_p = 1$ for all $w \in S_p$. In the first case, $h(v) \in L_p(1)h(\ddot{v})$ and $h(u) \in L_p(1)h(\ddot{u})$. So $h(v)h(\dot{u})$ and $h(u)h(\dot{v})$ belong to the same minimal left ideal $L_p(1)h(e_{p-1})$ in $g^{-1}(f(S_p))$. We have seen that these elements are idempotent, and so $h(u)h(\dot{v})h(v)h(\dot{u}) = h(u)h(\dot{v})$. Thus $h(u)h(v)h(u) = h(u)h(\dot{v})h(v)h(\dot{u})h(u) = h(u)h(\dot{v})h(u) = h(u)h(\ddot{u})h(\dot{v})h(u) = h(u)h(e_{p-1})h(u)$. This statement holds with v replaced by u, and so $h(u) = h(u)h(e_{p-1})h(u) = h(u)h(v)h(u)$.

Similarly, we can prove that $h(u)h(v)h(u) = h(u)$ if we assume that $w'_p = 1$ for all $w \in S_p$. $\square$

Recall that a *Bernside semigroup* $B(k,1,3)$ is the free semigroup on k generators in the variety of semigroups defined by the identity $x = x^3$. It is finite for every $k \in \mathbb{N}$ (see [2, Chapter 10, Theorem 3]).

The following theorem was proved in [9], but now we give more direct and short proof.

Theorem 2. *Let S be a finite band. If S is a coretract of a Bernside semigroup $B(k,1,3)$, then S is isomorphic to some semigroup of $\mathcal{P}$.*

Proof. Let $B = B(k,1,3)$ and let $f : B \to S$ be a coretraction. We may suppose that S is subsemigroup of B and that $f|_S = \mathrm{id}_S$. Let F be the free semigroup on a k-element alphabet A and let $h : F \to B$ be the canonical homomorphism. Observe that $h(u) = h(v)$ if and only if v can be obtained from u by a succession of operations in each of which a word w_1ww_2 is replaced by $w_1w^3w_2$, or vice versa. (Here words w_1, w, w_2 allowed to be empty.)

Let $w \in F$, $C \subseteq A$, and $\rho \subseteq C^2$. We shall use the following notation.

- $ct(w)$ is the set of letters in w.

Observe that $h(u)$ and $h(v)$ belong to the some completely simple component of B if and only if $ct(u) = ct(v)$ (see [2]).

- $w|_C$ is the word obtained from w by removing all letters in $A \setminus C$.
- $\alpha(w, C)$ is the first letter in $w|_C$.
- $\beta(w, C)$ is the last letter of $w|_C$.

Observe that if $h(u) = h(v)$, then $\alpha(u, C) = \alpha(v, C)$ and $\beta(u, C) = \beta(v, C)$.

- $\sigma(w, C, \rho)$ is the quantity of pairs of neighboring letters in $w|_C$ which belong to ρ.

Observe that if $h(u) = h(v)$, then $\sigma(u, C, \rho) \equiv \sigma(v, C, \rho)$ mod(2). To prove this, it suffices to consider the case $u = w_1ww_2$, $v = w_1w^3w_2$. Put $\sigma(t) = \sigma(t, C, \rho)$. Then

$$\sigma(v) = \begin{cases} \sigma(u) + 2\sigma(w) + 2 & \text{if } w|_C \neq \emptyset \text{ and } (\beta(w, C), \alpha(w, C)) \in \rho, \\ \sigma(u) + 2\sigma(w) & \text{otherwise.} \end{cases}$$

Lemma 2. *S is a chain of its rectangular components.*

Proof. Suppose the contrary. Then there exist $u, v \in h^{-1}(S)$ with $a \in ct(u) \setminus ct(v)$ and $b \in ct(v) \setminus ct(u)$. Put $\sigma(w) = \sigma(w, \{a,b\}, \{(a,b)\})$. Then $\sigma(uv) = 1$ and $\sigma(uvuv) = 2$, although $h(uvuv) = h(uv)$, a contradiction. $\square$

Let $S_1 > S_2 > \cdots > S_l$ be the rectangular components of S. Put

$$A_p = \{a \in A\colon\ fh(a) \in S_p\}.$$

Observe that for every $u \in h^{-1}(S)$, we have that $h(u) \in S_p$ if and only if $p = \max\{q \leqslant l\colon ct(u) \cap A_p \neq \emptyset\}$. Indeed, if $u = a_1 \cdots a_n$, then $h(u) = fh(u) = fh(a_1) \cdots fh(a_n)$.

Next, put

$$A_p^q = \bigcup_{r=p}^{q} A_r, \qquad S_p^q = \bigcup_{r=p}^{q} S_r \quad (p \leqslant q),$$
$$M_p = \bigl\{\alpha\bigl(u, A_p^l\bigr)\colon\ u \in h^{-1}\bigl(S_p^l\bigr)\bigr\}, \qquad N_p = \bigl\{\beta\bigl(u, A_p^l\bigr)\colon\ u \in h^{-1}\bigl(S_p^l\bigr)\bigr\}.$$

Observe that $M_p \cap A_p \neq \emptyset$ and $N_p \cap A_p \neq \emptyset$.

Lemma 3. *For every $p \in [1, l]$, one of the sets M_p, N_p is a singleton.*

Proof. Choose $u \in h^{-1}(S_l)$. Let $a = \alpha(u, A_p^l)$ and $b = \beta(u, A_p^l)$. Put $\sigma(w) = \sigma(w, A_p^l, \{(b, a)\})$. Since $\sigma(uu) = 2\sigma(u) + 1 \equiv \sigma(u) \pmod 2$, $\sigma(u)$ is odd. Suppose that there exist $v_1, v_2 \in h^{-1}(S_p^l)$ with $\alpha(v_1, A_p^l) \neq a$ and $\beta(v_2, A_p^l) \neq b$. Put $v = v_1v_2$. Since $\sigma(vv) = 2\sigma(v) \equiv \sigma(v) \pmod 2$, $\sigma(v)$ is even. Then $\sigma(uvu) = 2\sigma(u) + \sigma(v)$ is also even. On the other hand, in S, as in every chain of rectangular bands, the following statement holds true: if $x, z \in S_q$, $y \in S_r$, and $r \leqslant q$, then $xyz = xz$. Therefore $h(uvu) = h(uu) = h(u)$, and so $\sigma(uvu) \equiv \sigma(u) \pmod 2$, a contradiction. □

Lemma 4. *If $x \in S_p$, $y \in S_q$, $z \in S_r$, and $q \leqslant p, r$, then $xyz = xz$.*

Proof. It is convenient for us to adjoin identities $\emptyset$, $1_B = 1_S$ to F, B, S and to extend h, f in the obvious way. Put also $S_0 = \{1_S\}$. Then the lemma is obviously true if $q = 0$. Fix $q > 0$ and assume that the lemma holds for all smaller values of q. Take $u \in h^{-1}(x)$, $v \in h^{-1}(y)$, and $w \in h^{-1}(z)$. By Lemma 3, one of the sets M_q, N_q is a singleton. Suppose that $N_q = \{a\}$. Then we can write $u = u_1au_2$ and $v = v_1av_2$, where $ct(u_2), ct(v_2) \subseteq A_1^{q-1}$. Since $x = fh(u)$ and $y = fh(v)$, it follows from this that $x = x_1sx_2$ and $y = y_1sy_2$, where $s = fh(a) \in S_q$, $x_2 = fh(u_2)$, $y_2 = fh(v_2) \in S_0^{q-1}$, and $y_1 = fh(v_1) \in S_0^q$. So $xyz = x_1sx_2y_1sy_2z$ and $xz = x_1sx_2z$. It is clear that $sx_2y_1s = s$. By our inductive assumption, $sy_2z = sz$ and $sx_2z = sz$. Hence $xyz = x_1sz$ and $xz = x_1sz$. The case $|M_q| = 1$ is similar. □

We enumerate sets $M_p \cap A_p$ and $N_p \cap A_p$ as $\{a_{pi}\colon 1 \leqslant i \leqslant m_p\}$ and $\{b_{p\lambda}\colon 1 \leqslant \lambda \leqslant n_p\}$, respectively. Define functions ϕ_p and θ_p on S_p^l as follows. Let $x \in S_p^l$. Pick $u \in h^{-1}(x)$ and put

$$\phi_p(x) = \begin{cases} 0 & \text{if } \alpha\bigl(u, A_p^l\bigr) \notin A_p, \\ i & \text{if } \alpha\bigl(u, A_p^l\bigr) = a_{pi}, \end{cases} \qquad \theta_p(x) = \begin{cases} 0 & \text{if } \beta\bigl(u, A_p^l\bigr) \notin A_p, \\ \lambda & \text{if } \beta\bigl(u, A_p^l\bigr) = b_{p\lambda}. \end{cases}$$

We now define the map $\psi : S \to U$ putting for every $x \in S_p$,

$$\psi(x) = \phi_1(x)\phi_2(x)\cdots\phi_p(x)\theta_p(x)\theta_{p-1}(x)\cdots\theta_1(x).$$

It is clear that both $\phi_p(x) \neq 0$ and $\theta_p(x) \neq 0$. By Lemma 3, either $\phi_p(y) = 1$ for all $y \in S_p^l$ or $\theta_p(y) = 1$ for all $y \in S_p^l$.

To check that ψ is injective, let $x \in S_p$. Let $p_1 < p_2 < \cdots < p_s = p$ are all $r \in [1, p]$ with $\phi_r(x) \neq 0$, let $q_1 < q_2 < \cdots < q_t = p$ are all $r \in [1, p]$ with $\theta_r(x) \neq 0$, let $\phi_{p_j}(x) = i_j$, and let $\theta_{q_k}(x) = \lambda_k$. Pick $u \in h^{-1}(x)$. Then

$$u = a_{p_1 i_1} u_1 a_{p_2 i_2} u_2 \cdots u_{s-1} a_{p_s i_s} w b_{q_t \lambda_t} v_t \cdots v_2 b_{q_2 \lambda_2} v_1 b_{q_1 \lambda_1},$$

where $ct(u_j) \subseteq A_1^{p_j}$ and $ct(v_k) \subseteq A_1^{q_k}$. But then, by Lemma 4,

$$x = fh(a_{p_1 i_1} a_{p_2 i_2} \cdots a_{p_s i_s} b_{q_t \lambda_t} \cdots b_{\lambda_2 q_2} b_{q_1 \lambda_1})$$

and, consequently, x is uniquely determined by $\psi(x)$.

To see that ψ is homomorphism, let $x \in S_p$ and $y \in S_q$. It suffices to check that

(a) $\phi_r(xy) = \phi_r(x)$ if $r \leqslant p$;
(b) $\phi_r(xy) = \phi_r(y)$ if $p < r \leqslant q$;
(c) $\theta_r(xy) = \theta_r(y)$ if $r \leqslant q$;
(d) $\theta_r(xy) = \theta_r(x)$ if $q < r \leqslant p$.

Let $u \in h^{-1}(x)$, $v \in h^{-1}(y)$, and $w = uv$. If $r \leqslant p$, then $\alpha(w, A_r^l)$ occurs in u, because $ct(u) \cap A_p \neq \emptyset$, so $\phi_r(xy) = \phi_r(x)$. If $p < r \leqslant q$, then $\alpha(w, A_r^l)$ occurs in v, because $ct(v) \cap A_r^l = \emptyset$, so $\phi_r(xy) = \phi_r(y)$. The check of (c) and (d) is similar.

It remains to verify that the semigroup $\psi(S)$ satisfies condition (ii) in the definition of the class $\mathcal{P}$. Let $x \in S_p$ and let $\phi_q(x) = a \neq 0$ for some $q \in [1, p]$. Pick $u \in h^{-1}(x)$ and write it in the form $u = vaw$, where $ct(v) \subseteq A_1^{q-1}$. Define $y \in S_q$ by $y = fh(va)$. Since $x = fh(vaw)$, $yx = x$. By statement (a), $\phi_r(yx) = \phi_r(y)$ for all $r \leqslant q$. Hence $(\psi(y))'$ is the initial segment of $(\psi(x))'$. $\square$

From Theorem 2 and Trotter's theorem [4, Theorem 2.6] it follows the implication $(2) \Rightarrow (1)$. From Trotter's theorem we also deduce the next result.

Theorem 3. *Each weak projectives in $\mathcal{F}$ is a band.*

Proof. Let S be a weak projective in $\mathcal{F}$ and let $S = S_0 \supset S_1 \supset \cdots \supset S_n$ be its principal series. By Trotter's theorem, it suffices to prove that for every $i < n$, the quotient S_i/S_{i+1} is not a semigroup with zero multiplication. Assume the contrary and let m be the smallest such i. Take any integer $k > |\bigcup_{i=m+1}^{n} S_i|$. Let Q and T be cyclic monoids defined by relations $a^3 = a^2$ and $b^{k+2} = b^2$, respectively. Define homomorphisms $f : S \to Q$ and

$g : T \to Q$ by

$$f(x) = \begin{cases} a^0 & \text{if } x \in S_i \text{ with } i < m, \\ a & \text{if } x \in S_m, \\ a^2 & \text{if } x \in S_i \text{ with } i > m, \end{cases} \qquad g\big(b^j\big) = \begin{cases} a^0 & \text{if } j = 0, \\ a & \text{if } j = 1, \\ a^2 & \text{if } j > 1. \end{cases}$$

Then there is no a homomorphism $h : S \to T$ such that $g \circ h = f$, a contradiction. □

The implication $(4) \Rightarrow (1)$ follows from Theorems 2 and 3. It remains to prove $(7) \Rightarrow (8)$. For this, by Theorem 3, it suffices to prove the following proposition.

Proposition 1. *Let $\mathcal{S}$ be a category of semigroups containing all finite bands. Then in $\mathcal{S}$, every epimorphism into a finite band is surjective.*

Proof. Assume, by the contrary, there is a finite band S and a non-surjective epimorphism $f : T \to S$ in $\mathcal{S}$. Let J be a maximal component among all rectangular components I of S with $I \setminus f(T) \neq \emptyset$. Then either there exists $\mathcal{L}$-class L in J with $L \cap f(T) = \emptyset$ or there exists $\mathcal{R}$-class R in J with $R \cap f(T) = \emptyset$. Obviously, it suffices to consider the first case.

Define the equivalence θ_J on J by

$$\theta_J = \big\{(x, y) \in J^2\colon\ x\mathcal{L}y \text{ or both } L_x \cap f(T) \neq \emptyset \text{ and } L_y \cap f(T) \neq \emptyset\big\}.$$

It is partitioned J into the subset A being the union of $\mathcal{L}$-classes meeting $f(T)$, and the $\mathcal{L}$-classes disjoint $f(T)$. If $(x, y) \in \theta_J$ and $z \in J$, then $(xz, yz) \in \theta_J$ because $xz\mathcal{L}yz$, and $zx\mathcal{L}zy \in \theta_J$ because both $zx\mathcal{L}x$ and $zy\mathcal{L}y$. Hence θ_J is a congruence on J. For any another rectangular component I of S, put

$$\theta_I = \begin{cases} \nabla_I = I^2 & \text{if } I < J, \\ \Delta_I = \{(x, x)\colon\ x \in I\} & \text{otherwise.} \end{cases}$$

Define the equivalence θ on S by $\theta = \bigcup_I \theta_I$. Make sure that θ is congruence, that is for any $(x, y) \in \theta$ and $z \in S$, both $(zx, zy) \in \theta$ and $(xz, yz) \in \theta$. Obviously, only the case $(x, y) \in \theta_J$ and $z \in I > J$ needs the verification.

If $x\mathcal{L}y$, then $xz\mathcal{L}yz$ because $\mathcal{L}$ is a right congruence, and $zx\mathcal{L}zy$ because both $zx\mathcal{L}x$ and $zy\mathcal{L}y$. Let now $L_x \cap f(T) \neq \emptyset$ and $L_y \cap f(T) \neq \emptyset$. Then there are $a, b \in T$ such that $f(a) \in L_x$ and $f(b) \in L_y$. Since $I \subseteq f(T)$, there is $c \in T$ such that $f(c) = z$. But then $f(ac) = f(a)f(c) \in L_x z = L_{xz}$ and $f(bc) = f(b)f(c) \in L_y z = L_{yz}$. Hence, $L_{xz} \cap f(T) \neq \emptyset$ and $L_{yz} \cap f(T) \neq \emptyset$, and so $(xz, yz) \in \theta$. The fact that $(zx, zy) \in \theta$ is obvious because $L_{zx} = L_x$ and $L_{zy} = L_y$.

Let $\pi : S \to S/\theta$ is the canonical homomorphism. Define the homomorphism $\pi' : S \to S/\theta$ by

$$\pi'(x) = \begin{cases} \pi(x) & \text{if } x \in I \neq J, \\ A & \text{if } x \in J. \end{cases}$$

Then $\pi \circ f = \pi' \circ f$ and $\pi \neq \pi'$, a contradiction. □

Observe that as distinguished from the equivalence (6) $\Leftrightarrow$ (7), (3) $\Leftrightarrow$ (4) is a simple fact. It is a partial case of the following proposition.

Proposition 2. *Let $\mathcal{S}$ be a category of finite semigroups closed under finite direct product and subsemigroups. Then in $\mathcal{S}$, every weak absolute coretract is a weak projective.*

Proof. Let S be an absolute coretract in $\mathcal{S}$, let $f : S \to Q$ be a homomorphism, and let $g : T \to Q$ be a surjective homomorphism. We need to construct a homomorphism $h : S \to T$ with $g \circ h = f$.

Let I be the set of all maps $i : S \to R_i$, where $R_i = S$ or $R_i = T$. Define the injection $e : S \to \prod_I R_i$ by $e(x) = (i(x))_{i \in I}$. Let $X = e(S)$ and let F be the subsemigroup of $\prod_I R_i$ generated by X. Then each map $X \to R$, where $R = S$ or $R = T$ can be extended to a homomorphism $F \to R$.

Take any bijection $X \to S$ and extend it to a homomorphism $\alpha : F \to S$. Then, for every $x \in X$ choose $\gamma(x) \in g^{-1}(f \circ \alpha(x))$, and extend the map $X \ni x \mapsto \gamma(x) \in T$ to a homomorphism $\gamma : F \to T$. We have $g \circ \gamma = f \circ \alpha$. Since S is an absolute coretract, there exists a homomorphism $\beta : S \to F$ with $\alpha \circ \beta = \mathrm{id}_S$. Define $h : S \to T$ by $h = \gamma \circ \beta$. □

Let $\mathcal{F}G$ be a category of finite Clifford semigroups. From Hall's theorem it follows that both in $\mathcal{FR}$ and in $\mathcal{F}G$, weak projectives are projectives and weak absolute coretracts are absolute coretracts. By Theorems 1 and 2, in $\mathcal{FR}$, idempotent absolute coretracts are projectives. By Proposition 2, in $\mathcal{FG}$, all absolute coretracts are projectives.

Question 1. Is there an absolute coretract in $\mathcal{FR}$ other than bands?

Question 2. Is there a projective in $\mathcal{FG}$ other than bands?

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